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Mixing for positive operators in Banach lattices

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Abstract

In this paper we extend M. Lin's definition of mixing for positive contractions in $L_1(X,\Sigma,m)$ with $m(X)=1$ to positive operators in Banach lattices with weak-order units, and we generalize Lin's Theorem 2.1 (Z. Wahrsch. Verw. Gebiete 19 (1971) 231–249) to the case of power-bounded positive operators in *KB*-spaces. In the particular case of weakly compact power-bounded positive operators, the same theorem is extended to Banach lattices with order-continuous norms.

All Riesz spaces are assumed to be Archimedean.

Recall that an elements $e>0$ in a Riesz space is said to be a weak-order unit whenever the (principal) band $B(e)$ generated by $\{e\}$ satisfies $B(e)=E$.

A lattice norm $\|\cdot\|$ on a Riesz space is said to be order continuous if $(x_\alpha)\downarrow 0$ implies $\|x_\alpha\|\downarrow 0$. Standard examples of normed Riesz spaces with order-continuous norms are $L_p(\mu)$ $(1\leqslant p<\infty)$.

A Banach lattice (i.e., a complete normed Riesz space) E is said to be a *KB*-space if every norm-bounded increasing sequence in E is norm convergent. Examples of *KB*-spaces are furnished by l_1 and $L_1(\mu)$ (μ is the Lebesgue measure on $\mathbb{R}$) or, more generally, by all infinite-dimensional *AL*-spaces (i.e., Banach lattices with a norm which is additive on their positive cone; that is, $\|x+y\|=\|x\|+\|y\|$ for $y,x\geqslant 0$).

In [2], Lin obtained the following result for positive contractions in $L_1(x,\Sigma,m)$ with $m(X)=1$.

Theorem 0.1. *Let P be a Markov process with a finite invariant probability measure $\lambda\sim m$. Then the following condition are equivalent*:

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(a) *P is mixing.*
(b) *All weak*-limit points in L_∞ of $(P^n 1_A)_{n\in\mathbb{N}\cup\{0\}}$ are constants.*
(c) *For every $u \in L_1(\lambda)$ with $\int u\,d\lambda = 0$, $uP^n \to 0$ weakly in $L_1(\lambda)$.*
(d) *For every $v \in L_1(\lambda)$ and any increasing subsequence $(n_i)_{i\in\mathbb{N}}$,*

$$\left\| N^{-1}\sum_{n=1}^{N} vP^{ni} - \langle \lambda, v\rangle \right\|_1 \to 0.$$

Lin also extended the equivalence of (a) with (d) to $L_p(X, \Sigma, m)$, with $m(X) = 1$ and $1 \leqslant p < \infty$, in the same paper.

The aim of this paper is to prove the following (main) theorem:

Let E be a KB-space with a weak-order unit e and an order-continuous norm. Let E' be the dual of E with a weak-order unit e'. Let $T : E \to E$ with $Te \leqslant e$ be a power-bounded positive operator and let $T' : E' \to E'$ with $T'e' = e'$ be the adjoint of T. Then the following conditions are equivalent:

(a) *T' is mixing.*
(b) *All weak*-limit points in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in I(e')$.*
(c) *For every $u \in I(e)$ with $e'(u) = 0$, $\lim_{n\to\infty} x(T^n u) = 0$ for every $x \in I(e')$.*
(d) *For every $v \in I(e)$,*

$$\lim_{N\to\infty}\left\| N^{-1}\sum_{i=0}^{N} T^{ni} v - \big(e'(v)/e'(e)\big)e \right\|_E = 0$$

for every increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$.

We will start by defining mixing for positive operators in Banach lattices with weak-order units. Based on the Hopf decomposition defined by Zaharopol in [3], we will establish that, in the case of a Banach lattice E having a weak-order unit and an order-continuous norm, the above-mentioned definition is a necessary condition for T', a power-bounded positive operator in E', to be conservative. Then the key result of the paper will be discussed. That is, we will prove Theorem 1.3 in this paper. To do so, we will make use of two very important results, namely, the Abramovič–Wickstead theorem, or [1, Theorem 13.8, p. 206], which deals with the relative weak compactness of solid hulls of subsets of E, and [1, Lemma 12.15, p. 184], which allows us to obtain strong convergence from weak convergence in E.

After proving the main theorem, we will be concerned with its extension to more general Banach lattices by considering only weakly compact operators.

The terminology used in this paper can be found in the books of Aliprantis and Burkinshaw [1], Krengel [4], and Schaefer [5], and the paper of Zaharopol [6].

Definition 1.1. Let E be a Banach lattice with a weak-order unit e and let E' be its dual with a weak-order unit e'. Let $T : E \to E$ be a positive operator and let $T' : E' \to E'$ with

$T'e' \leqslant e'$ be its adjoint. Then T' is mixing if, for every $x \in [0, e']$, we have

$$\lim_{n\to\infty} \langle T^n u, x\rangle = \lim_{n\to\infty} \langle u, T'^n x\rangle = \bigl(x(e)/e'(e)\bigr)\langle u, e'\rangle$$

for every $u \in I(e)$ the (principal) ideal generated by $\{e\}$.

In accordance with the Hopf decomposition defined in [3], let I_{C} and I_{D} be the conservative ideal and the dissipative ideal of E generated by T, respectively, and let B_{C} and B_{D} be the dual conservative band and the dual dissipative band of E' generated by T', respectively.

Definition 1.2. Let E be a Banach lattice and let E' be its dual. Let $T : E \to E$ be a power-bounded positive operator and let $T' : E' \to E'$ be its adjoint. Suppose B_{C} and B_{D} are defined as above. Then T' is said to be conservative if $B_{\mathrm{D}} = \{0\}$.

For our very next discussion we need to recall the following results from [3].

Let E, E', T', e' be as in Definition 1.1. Suppose that E has an order-continuous norm and $e' = e'_{\mathrm{C}} + e'_{\mathrm{D}}$, where $e'_{\mathrm{C}} = P_{B_{\mathrm{C}}}(e')$ and $e'_0 = P_{B_{\mathrm{D}}}(e')$. Then $B_{\mathrm{C}}(e') = B_{\mathrm{C}}$, $B_{\mathrm{D}}(e') = B_{\mathrm{D}}$, and $E' = B_{\mathrm{C}}(e') \oplus B_{\mathrm{D}}(e')$, where $B_{\mathrm{C}}(e')$ and $B_{\mathrm{D}}(e')$ are (principal) bands generated by $\{e'_{\mathrm{C}}\}$ and $\{e'_{\mathrm{D}}\}$, respectively.

We are now in a position to prove the following proposition.

Proposition 1.1. *Let E be a Banach lattice with a weak-order unit e and an order-continuous norm. Let E' be its dual. Suppose $T : E \to E$ is a power-bounded positive operator and let $T' : E' \to E'$ with $T'e' \leqslant e'$ be its adjoint. If T' is mixing then T' is conservative.*

Proof. We shall divide the proof into two steps.

Step 1. We first show that Definition 1.1 can be extended to the entire Banach lattice E. Let $w \in E$, $x \in [0, e']$, and $\varepsilon > 0$. Since E has an order-continuous norm, it is clear that there exists an element $u \in (e)$ such that $\|w - u\|_E < \varepsilon/2M$, where $M > 0$, and $M \geqslant \sup_n\{\|T'^n x\|_{E'} + (x(e)/e'(e))\|e'\|_{E'}\}$. Obviously, we have

$$\begin{aligned}\bigl|\bigl\langle w, T'^n x - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr| &\leqslant \bigl|\bigl\langle w - u, T'^n x - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr| \\ &\quad + \bigl|\bigl\langle u, T'^n x - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr|.\end{aligned}$$

It is clear that

$$\bigl|\bigl\langle w - u, T'^n - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr| \leqslant \|w - u\|_E \bigl\|T'^n x - \bigl(x(e)/e'(e)\bigr)e'\bigr\|_{E'} \leqslant \varepsilon/2.$$

By Definition 1.1, for n large enough,

$$\bigl|\bigl\langle u, T'^x - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr| < \varepsilon/2.$$

Hence there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\bigl|\bigl\langle w, T'^n x - \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle\bigr| < \varepsilon.$$

Step 2. We now prove the proposition. Since by the observation above $B_{\mathrm{D}} = B(e'_{\mathrm{D}})$, we will show that $B(e'_{\mathrm{D}}) = \{0\}$ instead. Suppose that $B(e'_{\mathrm{D}}) \neq \{0\}$. Then $e'_{\mathrm{D}} \neq 0$ and by the characterization of elements of B_{D} given by [3, p. 36] there exist $z \in E'$, $0 \leqslant z \leqslant e'_{\mathrm{D}}$, and $u \in E$, $u \geqslant 0$, such that $0 < \sum_{n=0}^{\infty} \langle T^n u, z \rangle < +\infty$. It follows then that $\lim_{n\to\infty} \langle T^n u, z \rangle = 0$. Since $z \in [0, e']$, $\lim_{n\to\infty} \langle T^n u, z \rangle = \langle z(e)/e'(e) \rangle \langle u, e' \rangle$, by the conclusion of Step 1. So, obviously, we have $(z(e)/e'(e))\langle u, e' \rangle = 0$, or, equivalently, $z(e) = 0$ with $z \geqslant 0$ (1) or $\langle u, e' \rangle = 0$ with $u \geqslant 0$ (2).

We first show that condition (2) cannot hold. Assume that it does. By [5, Corollary 1, p. 90] u is order continuous. Therefore C_u, the carrier of u, is different from $\{0\}$ since $u \neq 0$ by the definition of B_{D}. Let $x \in C_u$, $x \neq 0$. Then $x \wedge e' = 0$ since $e' \in N_u$, the null ideal of u, by virtue of $\langle u, e' \rangle = 0$. But this is impossible since e' is a weak-order unit of E'.

To conclude this discussion, we need to show that only $z = 0$ satisfies condition (1) above. Then e'_{D} will be equal to 0 and so we will get a contradiction to the assumption that $B(e'_{\mathrm{D}}) \neq \{0\}$. Hence $B(e'_{\mathrm{D}})$ would have had to be trivial in the first place.

Suppose $z(e) = 0$, with $z \geqslant 0$ and $z \neq 0$. Then, obviously, $I(e) \subseteq N_z$, where $N_z = \{v \in E\colon z(|v|) = 0\}$. To see that $B(e)$ is included in N_z, let $u \in B(e)$, $u > 0$, and $u \notin I(e)$. Then $u = \sup_m \{u_m\}$, where $u_m = u \wedge (me)$, for each natural number m. The non-negative real sequence $(\langle u - u_m, z \rangle)_{m\in\mathbb{N}}$ clearly decreases to 0, since E has an order-continuous norm and therefore $\lim_{m\to\infty} \langle u_m, z \rangle = \langle u, z \rangle$. So $\langle u, z \rangle = 0$, since $u_m \in I(e)$ for each $m \in \mathbb{N}$ and $I(e) \subset N_z$. Hence $E \subset N_z$. It is then clear that $\langle T^n u, z \rangle = 0$, $\forall n \in \mathbb{N}$, and so $\sum_{n=0}^{\infty} \langle T^n u, z \rangle = 0$, which contradicts the choice of u and z. □

Definition 1.3. Let E be a Riesz space with a weak-order unit e. An element x of E is said to be a constant if $x = \alpha e$ for some $\alpha \in \mathbb{R}$.

The next theorem offers two conditions equivalent to the mixing.

Theorem 1.1. *Let E, E', T, T', e, and e' be as in Definition* 1.1. *Suppose that E has an order-continuous norm and $Te = e$. Then the following conditions are equivalent*:

(a) *T' is mixing.*
(b) *All weak*-limit points in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in I(e')$.*
(c) *For every $u \in I(e)$ with $e'(u) = 0$, $x(T^n u) \to 0$ as $n \to \infty$ for every $x \in I(e')$.*

Proof. Here we will mostly use the ideas and arguments of the proof of Theorem 2.1 in [2].

(a) ⇒ (b) Since E has an order-continuous norm, Definition 1.1 or condition (a) can be extended to the entire Banach lattice E; i.e., for every $x \in [0, e']$,

$$\lim_{n\to\infty} \langle u, T'^n x \rangle = \bigl(x(e)/e'(e)\bigr)\langle u, e' \rangle = \bigl\langle u, \bigl(x(e)/e'(e)\bigr)e' \bigr\rangle$$

for every $u \in E$. As one can see, the latter limit holds by virtue of the conclusion of Step 1 in the proof of Proposition 1.1. Now suppose that, for $x \in (c')$, $x \geqslant 0$, $\lim_{n\to\infty} \langle T^n u, x \rangle = (x(e)/e'(e))\langle u, e' \rangle$ for every $u \in E$. Then from the decomposition $x = x^+ - x^-$, where, obviously, $x^+, x^- \in I(e')$, it follows that, for every $x \in I(e')$,

$$\lim_{n\to\infty} \langle T^n u, x \rangle = \bigl(x(e)/e'(e)\bigr)\langle u, e' \rangle$$

by the linearity of e and each $T^n u$ as elements of E''. So, in order to show that the conclusion of Step 1 from the proof of Proposition 1.1 remains true for every $x \in I(e')$, it is enough to consider only positive x's. Assume (a) and let $x \in I(e')$, $x \geqslant 0$. Then there exists $\lambda \in \mathbb{R}^+$, $\lambda \neq 0$, such that $0 \leqslant x \leqslant \lambda e'$. So $0 \leqslant (x/\lambda) \leqslant e'$, and hence from

$$\lim_{n\to\infty} \langle T^n u, x\rangle = \lim_{n\to\infty} \lambda \langle T^n u, x/\lambda\rangle$$

and Definition 1.1 we get

$$\lim_{n\to\infty} \langle u, T'^n x\rangle = \bigl\langle u, \bigl(x(e)/e'(e)\bigr)e'\bigr\rangle$$

for every $u \in E$ and $x \in I(e')$, or, equivalently, for each $x \in I(e')$, $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ weak*-converges to $x(e)/e'(e)e'$. This is also saying that, for each $x \in I(e')$, the sequence $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ has a unique weak*-limit point which is therefore equal to $(x(e)/e'(e))e'$. Hence (b) follows.

(b) $\Rightarrow$ (c) It is enough to show that $\lim_{n\to\infty}\langle T^n u, x\rangle = 0$ for every $x \in [0, e']$ and $u \in I(e)$ with $e'(u) = 0$. Assume (b) and suppose that (c) is false; i.e., for some $v \in I(e)$, with $e'(v) = 0$, there exists $x \in [0, e']$ such that $x(T^n v)$ does not converge to 0 as $n \to \infty$. Then we can find a strictly increasing subsequence $(n_i)_{i\in\mathbb{N}}$ and $\varepsilon > 0$ such that $\langle v, T'^{ni} x\rangle \geqslant \varepsilon$, $\forall i \in \mathbb{N}$. If $y \in E'$ is a weak*-limit point of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$, it is a constant by (b), i.e., $y = \alpha e'$ for some $\alpha \in \mathbb{R}$, and so $\langle v, y\rangle = \alpha\langle v, e'\rangle = 0$. But $U = \{z \in E'\colon |\langle v, z\rangle| < \varepsilon\}$ is a weak*-open neighborhood of y, so $(T'^{ni} x) \in U$ for some i. Thus we obtain a contradiction to the assumption: $\langle v, T'^{ni} x\rangle \geqslant \varepsilon$, $\forall i \in \mathbb{N}$. Hence (c) is true.

(c) $\Rightarrow$ (a) Since $Te = e$, $\forall x \in I(e')$, we have $\langle e, T'^n x\rangle = \langle T^n e, x\rangle = \langle e, x\rangle$.

Let $v \in I(e)$ and define $u = v - (e'(u)/e'(e))e$. Obviously, $u \in I(e)$ and $e'(u) = 0$. We then have

$$\begin{aligned}\langle v, T'^n x\rangle - \bigl(x(e)/e'(e)\bigr)\langle v, e'\rangle &= \langle v, T'^n x\rangle - \bigl(e'(v)/e'(e)\bigr)\langle e, x\rangle \\ &= \langle v, T'^n x\rangle - \bigl(e'(v)/e'(e)\bigr)\langle e, T'^n x\rangle\end{aligned}$$

by the observation made in the beginning of this proof. Hence

$$\begin{aligned}\langle v, T'^n x\rangle - \bigl(x(e)/e'(e)\bigr)\langle v, e'\rangle &= \bigl\langle v - \bigl(e'(v)/e'(e)\bigr)e, T'^n x\bigr\rangle \\ &= \bigl\langle T^n\bigl(v - \bigl(e'(v)/e'(e)\bigr)e\bigr), x\bigr\rangle \\ &= x\bigl(T^n\bigl(v - \bigl(e'(v)/e'(e)\bigr)e\bigr)\bigr) \\ &= x(T^n u).\end{aligned}$$

So $\langle v, T'^n x\rangle - (x(e)/e'(e))\langle v, e'\rangle \to 0$ as $n \to \infty$, since $\lim_{n\to\infty} x(T^n u) = 0$ by (c). Thus T' is mixing. □

A sufficient condition for the operator T' to be mixing is presented next.

Theorem 1.2. *Let E be a Banach lattice with a weak-order unit e and E' be its dual with a weak-order unit e'. Let $T : E \to E$ be a power-bounded positive operator and let $T' : E' \to E'$ with $T'e' \leqslant e'$ be its adjoint. Then T' is mixing if, for every $v \in I(e)$ and increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$,*

$$\lim_{N\to\infty} \Biggl\| N^{-1} \sum_{i=1}^{N} T^{ni} v - \bigl(e'(v)/e'(e)\bigr)e \Biggr\|_E = 0.$$

Proof. Assume that $\lim_{n\to\infty}\|N^{-1}\sum_{i=1}^{N}T^{n_i}v-(e'(v)/e'(e))e\|=0$ for every increasing subsequence $(n_i)_{i\in\mathbb{N}}$, and $v\in I(e)$. Note that, by setting $n_i=i$, one can see that $(e'(v)/e'(e))e$ is a fixed point of T by the power boundedness of T. To show that, for every $x\in[0,e']$, $\lim_{i\to\infty}\langle T^{ni}v-(e'(v)/e'(e))e,x\rangle=0$ for every increasing subsequence $(n_i)_{i\in\mathbb{N}}$, assume $\lim_{i\to\infty}\langle T^{ni}v-(e'(v)/e'(e))e,x\rangle\neq 0$ for some $x\in[0,e']$ and some increasing subsequence $(n_i)_{i\in\mathbb{N}}$. Then, for those x and $(n_i)_{i\in\mathbb{N}}$, it must follow that $\lim_{N\to\infty}\langle N^{-1}\sum_{n=1}^{N}T^{ni}(v-e'(v)/e'(e)e),x\rangle\neq 0$ by the Cesàro theorem, and so we get a contradiction to the strong convergence to 0 of $(N^{-1}\sum_{i=1}^{N}T^{ni}v-(e'(v)/e'(e))e)_{N\in\mathbb{N}}$ for every increasing subsequence $(n_i)_{i\in\mathbb{N}}$.

Hence $(T^{n_i}v-(e'(v)/e'(e)e))_{i\in\mathbb{N}}$ converges weakly to 0 for every increasing subsequence $(n_i)_{i\in\mathbb{N}}$, and so necessarily we have

$$\lim_{n\to\infty}\bigl\langle T^n v-\bigl(e'(v)/e'(e)\bigr)e,x\bigr\rangle$$

for every $x\in[0,e']$ and $v\in I(e)$. □

Lemma 1.1. *Let E be a KB-space with a weak-order unit e and let $T:E\to E$, $Te\leqslant e$, be a power-bounded positive operator. Let E' be the dual of E with a weak-order unit e' and let $T':E'\to E'$ with $T'e'=e'$ be the adjoint of T. Suppose that, for every $u\in E$, $e'(u)=0$, $\lim_{n\to\infty}x(T^nu)=0$ for every $x\in E'$ (!). Then there exist a (strictly) increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$ and a constant $z=\alpha e$ in E such that $\lim_{i\to\infty}\langle|T^{ni}u|-z,x\rangle=0$ for every $x\in E'$ and $u\in E^0=\{w\in E\colon e'(u)=0\}$.*

Proof. We shall divide the proof into several steps.

Step 1. We show that e' is strictly positive on $[0,e]$. Let $u\in[0,e]$, $u\neq 0$, and suppose $e'(u)=0$. Since E has an order-continuous norm, by [5, Corollary 1, p. 90], $E\subseteq(E')_n$, where $(E')/n$ is the band of E'' generated by E and consisting of all order-continuous linear forms on E'. It is then clear that C_u, the carrier of u, is not trivial. Let $x\in E'$, $x\neq 0$, be in C_u. Then, by virtue of $e'(u)=0$, $x\wedge e=0$, which is impossible since e is a weak-order unit. Hence $e'(u)>0$ for every $u\in[0,e]$, $u\neq 0$.

Step 2. Under the assumptions of the lemma, $Te=e$. In fact, by virtue of $Te\leqslant e$, the order completeness of E, the order continuity of the norm of E, and $0\leqslant(e-T^ne)\leqslant e$, the sequence $(T^ne)_{n\in\mathbb{N}}$ converges strongly to some $v\in[0,e]$. By $T'e'\leqslant e'$, this identity

$$0<\langle e,e'\rangle=\bigl|\langle e,T'^{\,n}e'\rangle\bigr|=\bigl|\langle T^ne,e'\rangle\bigr|\leqslant\|T^ne\|_E\|e'\|_{E'},$$

and the power boundedness of T, v is a non-zero fixed point of T. Let $w=v-(e'(v)/e'(e))e$. Then, by the assumption (!) of the lemma, $\lim_{n\to\infty}\langle T^mw,x\rangle=0$ for every $x\in E'$. This implies that

$$\begin{aligned}0=\lim_{n\to\infty}\langle T^nw,x\rangle&=\lim_{n\to\infty}\bigl\langle T^nv-\bigl(e'(v)/e'(e)\bigr)T^ne,x\bigr\rangle\\&=\langle v,x\rangle-\bigl(e'(v)/e'(e)\bigr)\lim_{n\to\infty}\langle T^ne,x\rangle\\&=\langle v,x\rangle-\bigl(e'(v)/e'(e)\bigr)\langle v,x\rangle\end{aligned}$$

for every $x\in E'$, which obviously suggests that either $v=e$ (1) or $v\neq e$ and $e'(v)=e'(e)$ (2), since $v\neq 0$. We now need only to show that condition (2) cannot hold. Assume that it does hold. Then, in view of the conclusion of Step 1 and the obvious inequality, $0 \leqslant e - v \leqslant e$, we obtain $e'(e) > e'(v)$, which contradicts condition (2) at once. So $v = e$ and hence $Te = e$.

Step 3. We now prove the lemma. Here we will use some of the arguments of [5, Lemma 8.6, p. 347]. To this end let $u \in E^0$. Then $\lim_{n\to\infty} x(T^n u) = 0$ for $x \in E'$ by our assumption. So $\{T^n u\colon n = 0, 1, \ldots\}$ is $\sigma(E, E')$-relatively compact in E, by [1, Corollary 10.16, p. 159], and hence $\{|T^n u|\colon n = 0, 1, \ldots\}$ is $\sigma(E, E')$-relatively compact by the Abramovič–Wickstead theorem or [1, Theorem 13.8, p. 206]. Denote by $B \subset E_+$ the set of weak-limit points of the sequence $(|T^n u|)_{n\in\mathbb{N}\cup\{0\}}$ and let $B_0 \subset B$ be the subset of those $z \in B$ for which $|||z||| = \inf_{y\in B} |||y|||$, where $|||u||| = \sup_k\{\|T^k u\|\}$ for $u \in E$; B_0 is weakly compact. We claim that $TB_0 \subset B_0$. In fact, if $z \in B_0$ then z is the $\sigma(E, E')$-limit point of some subsequence $(|T^{n_i} u|)_{i\in\mathbb{N}}$; if $y \in B$ is a limit point of the sequence $(|T^{(n_i)+1} u|)_{i\in\mathbb{N}}$ then $y \leqslant Tz$. Hence, since T is a contraction for $|||\cdot|||$, we have $y \in B_0$ and it follows that $y = Tz$; therefore $TB_0 \subset B_0$.

Consider $w \in B_0$ and $(T^n w)_{n\in\mathbb{N}\cup\{0\}}$. Then, in view of $TB_0 \subset B_0$ and weak compactness of B_0, by [1, Theorem 10.13, p. 154], there exist $w_0 \in B_0$ and a subsequence of natural number $(n_i)_{i\in\mathbb{N}}$ such that $w_0 = \text{weak-}\lim(T^{ni} w)$. Set $z = w - (e'(w)/e'(e))e$. Then weak-$\lim(T^{ni} z) = 0 = w_0 - (e'(w)/e'(e))e$, or, equivalently, $w_0 = (e'(w)/e'(e))e$. Thus some (increasing) subsequence of $(|T^n u|)_{n\in\mathbb{N}\cup\{0\}}$ converges to w_0 weakly. □

The next theorem provides a sufficient condition for strong convergence of averages of the iterates of power-bounded positive operators in *KB*-spaces.

Theorem 1.3. *Let E be a KB-space with a weak-order unit e and let E' be its dual with a weak-order unit e'. Let $T: E \to E$ with $Te \leqslant e$ be a power-bounded positive operator and let $T': E' \to E'$ with $T'e' \leqslant e'$ be its adjoint. Suppose*

(c_0) *all weak*-limit points in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in E'$;*
(c_1) *T has a non-zero fixed point.*

Then, for every $v \in I(e)$,

$$\lim_{N\to\infty} \left\| N^{-1} \sum_{i=1}^{N} T^{n_i} v - \big(e'(v)/e'(e)\big)e \right\|_E = 0$$

for every increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$.

Proof. We shall divide the proof into several steps.

Step 1. $Te = e$. Let $v \neq 0$ be a fixed point of T. Let us first establish that it is enough to consider only a positive v. From the decomposition $v = v^+ - v^-$, we have $v^+ - v^- = Tv = T(v^+) - T(v^-)$. By [5, Corollary 2, p. 52], $v^+ = (T(v^+ - v^-))^+$ and $v^- = (T(v^+ - v^-))^-$. Clearly it follows that

$$\begin{aligned} v^+ &= \big(T(v^+ - v^-)\big)^+ = T(v^+ - v^-) \vee 0 = T(v^+ - v^-) \vee T(0) \\ &\leqslant T\big((v^+ - v^-) \vee 0\big) = Tv^+, \end{aligned}$$

$$v^- = \big(T(v^+ - v^-)\big)^- = -\big(T(v^+ - v^-)\big) \vee 0 = T\big(-(v^+ - v^-)\big) \vee 0$$
$$\leqslant T\big(-(v^+ - v^-) \vee 0\big) = Tv^-.$$

Now consider the sequence $(T^n v^+)_{n\in\mathbb{N}\cup\{0\}}$. It is increasing and norm bounded. Therefore it is strongly convergent and its limit is a fixed point of T, by the boundedness of T. Denote that limit by v_∞^+. Similarly we obtain v_∞^- as the strong limit of the sequence $(T^n v^-)_{n\in\mathbb{N}\cup\{0\}}$. From $v = T^n v = T^n(v^+) - T^n(v^-)$, it is clear that $v = v_\infty^+ - v_\infty^-$, where $v_\infty^+ \geqslant 0$, $v_\infty^- \geqslant 0$, $Tv_\infty^+ = v_\infty^+$, and $Tv_\infty^- = v_\infty^-$. Hence it is enough to consider only a positive fixed point v.

By virtue of $T'e' \leqslant e'$, the order continuity of the norm of E, and [5, Corollary, p. 891], $(T'^n e')_{n\in\mathbb{N}\cup\{0\}}$ converges strongly. Therefore, from

$$0 < \langle v, e'\rangle = \lim_{n\to\infty} \langle T^n v, e'\rangle = \lim_{n\to\infty} \big|\langle v, T'^n e'\rangle\big| \leqslant \|v\|_E \lim_{n\to\infty} \|T'^n e'\|_E,$$

we get $\lim(T'^n e') \neq 0$. By our assumption we have $\lim(T'^n e') = \alpha e'$ for some $\alpha \in \mathbb{R}^+$ and hence $T'e' = e'$ by the boundedness of T. Theorem 1.1 and the assumption (c_0) of the lemma clearly imply that, for every $u \in E$ with $e'(u) = 0$, $\lim_{n\to\infty} x(T^n u) = 0$ for every $x \in E'$, and so $Te = e$ by Step 1 of the proof of Lemma 1.1.

Step 2. $(|T^n u|)_{n\in\mathbb{N}\cup\{0\}}$ weakly converges to 0 for every $u \in E^0$ of the form $w - (e'(w)/e'(e))e$ with $w \in I(e)$. Assume that it does not. Then B, the set of all weak-limit points of $(|T^n u|)_{n\in\mathbb{N}\cup\{0\}}$, is different from $\{0\}$ and so, by Lemma 1.1 and [1, Corollary 10.16, p. 159], for every $x \in E'$, $\lim_{n\to\infty e}\langle |T^{n_i} u| - \alpha e, x\rangle = 0$ for some increasing subsequence $(n_i)_{i\in\mathbb{N}}$ and $\alpha \in \mathbb{R}^+\backslash\{0\}$, taking into account the assumption (c_0) of the theorem. Now consider $T^{n_i}u$ for a fixed i.

If $T^{n_i}u \ngeqslant 0$ and $T^{n_i}u \nleqslant 0$ then $c_{(T^{n_i}u)^+} \neq \{0\}$ and $C_{(T^{n_i}u)^-} \neq \{0\}$, where $C_{(T^{n_i}u)^+}$ and $C_{(T^{n_i}u)^-}$ are carriers of $(T^{n_i}u)^+$ and $(T^{n_i}u)^-$, respectively. By [1, Theorem 5.2, p. 56], there exists $x \in C_{(T^{n_i}u)^+}$, $x \geqslant 0$, such that $x((T^{n_i}u)^-) = 0$ for each i, keeping in mind that E has an order-continuous norm. We denote such an element x by x_{n_i}. Now let $(x_{n_i})_{i\in\mathbb{N}}$ be the sequence of positive elements of E such that, for each i, $\langle T^{n_i}u, x_{n_i}\rangle = \langle |T^{n_i}u|, x_{n_i}\rangle$ (!). By the choice of the subsequence $(n_i)_{i\in\mathbb{N}}(|T^{n_i}u| - \alpha e)_{i\in\mathbb{N}}$ converges weakly to 0. That is, given $\varepsilon > 0, \exists N, \forall i \geqslant N, |\langle |T^{n_i}u| - \alpha e, x_i\rangle| < \varepsilon$ for every $j \in \mathbb{N}$. Set $U_k = \{u\colon |x_{n_k}(u)| < \varepsilon\}$ with $x_{n_k} \in C_{(T^{n_k}u)^+}$.

Let $K = \{k\colon k \in I \subseteq \mathbb{N}\}$ be the set of natural numbers such that for each k there is only a finite number of $T^{n_l}u$'s whose carriers have non-trivial intersection with the carrier of $T^{n_k}u$.

Case 1. Suppose K is infinite. Then $\forall\varepsilon > 0$, $\exists N = N(\varepsilon, k)$ such that $T^{n_i}u \in U_k$ and $T^{n_k}u \wedge T^{n_i}u = 0$, $\forall i \geqslant N$. Consider $T^{n_N}u$ and choose s such that $T^{n_N}u \wedge T^{n_i}u = 0$, $\forall i \geqslant s$. Then we will obtain an order-bounded disjoint subsequence of $(T^{n_i}u)_{i\in\mathbb{N}}$ whose strong limit will obviously be equal to 0 by [1, Theorem 12.13, p. 183], which is a contradiction to the weak convergence of $(|T^{n_i}u|)_{i\in\mathbb{N}}$ to αe.

Case 2. Suppose $\mathbb{N}\backslash K$ is infinite and let $j \in \mathbb{N}\backslash K$ be such that there is only a finite number of $(T^{n_l}u)^+$'s and $(T^{n_l}u)^-$'s whose carriers have non-trivial intersection with the carrier of $(T^{n_j}u)^+$ and $(T^{n_j}u)^-$, respectively. If the number of such j's is infinite then as in the first case we get a contradiction.

Now let $k \in \mathbb{N}\backslash K$ be such that the carrier of $(T^{n_k}u)^+$ has a non-trivial intersection with the carriers of infinitely many $(T^{n_l}u)^+$'s. Without loss of generality we can assume that there are infinitely many of these k's. Let $\varepsilon_0 > 0$ be such that $x_{n_k}(\alpha e) - \varepsilon_0 = c > 0$. Then $\forall 0 < \varepsilon \leqslant \varepsilon_0$, $\exists N$, such that, for every $n \geqslant N$, $\exists l \geqslant n$, with

$$c < x_{n_k}(\alpha e) - \varepsilon < x_{n_k}(T^{n_l}u) < x_{n_k}(\alpha e) + \varepsilon, \tag{\#}$$

taking into account that $x_{n_k}(T^{n_1}u) = x_{n_k}(|T^{n_1}u|)$, by virtue of $x_{n_k} \in C_{(T^{n_k}u)^+} \cap C_{(T^{n_1}u)^+}$.

To pursue our discussion we need to show that C_{n_k}, the carrier of x_{n_k} is not trivial in E. In fact, since $x_{n_k} \in C_{(T^{n_k}u)^+}$, with $(T^{n_k}u)^+ \neq 0$, if E is included in N_{n_k}, the null ideal of x_{n_k}, then $x_{n_k}((T^{n_k}u)^+) = 0$, which is a contradiction to the fact that $x_{n_k} \in C_{(T_u^{n_k})^+}$. Hence $C_{n_k} \neq \{0\}$. Now we can also claim that $x_{n_k}(e) \neq 0$. Let $x_{n_k}(e) = 0$. Since $C_{n_k} \neq \{0\}$, there exists $v \in C_{n_k}$, $v \neq 0$, and then $e \wedge |v| = 0$. But this is impossible since e is a weak-order unit of E. Hence $x_{n_k}(e) \neq 0$ for each k. So, by virtue of (#), $(T^{n_i}u)_{i\in\mathbb{N}}$ cannot be in U_k, a weak neighborhood of 0, which contradicts the weak convergence to 0 of $(T^n u)_{n\in\mathbb{N}\cup\{0\}}$. Hence $(|T^{n_i}u|)_{n_i\in\mathbb{N}}$ had to converge weakly to 0 in the first place.

If $T^n u \leqslant 0$ or $\geqslant 0$ for some n then $|T^k u| = -T^k u$ or $T^k u$ for all $k \geqslant n$, and so weak convergence to 0 of $(|T^n u|)_{n\in\mathbb{N}\cup\{0\}}$ follows at once from weak convergence to 0 of $(T^n u)_{n\in\mathbb{N}\cup\{0\}}$.

Step 3. We now prove the theorem. The sequence $(\langle |T^n u|, e'\rangle)_{n\in\mathbb{N}\cup\{0\}}$ is decreasing and bounded below by 0. Therefore it converges to 0, by virtue of the weak convergence to 0 of the subsequence $(|T^{n_i}u|)_{i\in\mathbb{N}}$.

We already established in Steps 1 and 2 that $Te = e$ and that, for $u \in E$, $u = w - (e'(w)/e'(e))e$ with $w \in I(e)$, $\lim_{n\to\infty}\langle |T^n u|, e'\rangle = 0$. Before proceeding, notice that $u \in I(e)$, $e'(u) = 0$, and there exists λ such $|u| \leqslant \lambda e$. Therefore, from

$$0 \leqslant \bigl(|T^n u|/\lambda\bigr) \leqslant \bigl(T^n(|u|)/\lambda\bigr) \leqslant e,$$

it follows that $|T^n u|/\lambda$ belongs to $[0, e]$ for each n. Hence, by [1, Lemma 12.15, p. 184],

$$0 = \lim_{n\to\infty} \bigl\| |T^n u|/\lambda \bigr\|_E = \lim_{n\to\infty} \|T^n u/\lambda\|_E$$

for every $u \in (e)$, $e'(u) = 0$, taking into account the conclusion of Step 1 in the proof of Lemma 1.1. Now, by setting $u = v - (e'(v)/e'(e))e$ for each $v \in I(e)$ we have

$$\lim_{N\to\infty} \left\| N^{-1} \sum_{n=1}^{N} T^{n_i} v - \bigl(e'(v)/e'(e)\bigr)e \right\|_E = 0$$

for every increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$, by the Cesàro theorem. $\square$

Proposition 1.2. *Let E be a Banach lattice with a weak-order unit e and an order-continuous norm. Let E' be its dual with a weak-order unit e'. If $y \in E'$, $y \geqslant 0$, is not a constant then there exists $u \in E$, $u \neq 0$, such that $e'(u) = 0$ and $y(u) > 0$.*

Proof. Let $y \in I(e')$, $y \geqslant 0$, and $\alpha \in \mathbb{R}^+$. Define $A_\alpha = \{v \in E^+\colon\ y(v) > \alpha e'(v)\}$ and $B_\alpha = \{v \in E^+\colon\ y(v) \leqslant \alpha e'(v)\}$. Assume that y is not a constant. We first show that there exists α such that $A_\alpha \neq \emptyset$. Let $y \in I(e')$. Then there exists $\lambda \geqslant 0$ such that $y \leqslant \lambda e'$. It is clear that $B_\lambda = E^+$ and so $A_\lambda = \emptyset$. Set $\alpha = \lambda/2$. If $A_\alpha \neq \emptyset$, then $\lambda/2$ is one of those α's we need; otherwise we set $\alpha = \lambda/2^2$. Again, if $A_\alpha \neq \emptyset$, then $\lambda/2^2$ is one of those α we need; otherwise we set $\alpha = \lambda/2^3$ and repeat the same analysis over and over. But this process must end since otherwise $0 \leqslant y(v) \leqslant (\lambda/2^k)e'(v)$ for every $v \in E^+$ and $k \in \mathbb{N}$ would have implied that $0 \leqslant y \leqslant (\lambda/2^k)e'$ for every $k \in \mathbb{N}$. Since the Banach lattice E' is Archimedean, the latter inequalities clearly suggest y is equal to 0. But that is a contradiction to the fact that y is not a constant. So there must exist $k \in \mathbb{N}$ for which $\alpha = \lambda/2^k$ with $(\alpha e' - y)^- \neq 0$. By combining [1, Theorem 12.9, p. 179] and [5, Theorem 5.10, p. 89], we can see that $(\alpha e' - y)^-$ is order continuous. Therefore it has a non-zero carrier in E. Hence $\{0\} \neq A_\alpha \neq \emptyset$. Let $w \in A_\alpha$, $w \neq 0$, and $z \in B_\alpha$, $z \neq 0$, and define $u = (w/e'(w)) - (z/e'(z))$. It is clear that $e'(u) = 0$. But

$$\begin{aligned} y(u) &= \big(y(w)/e'(w)\big) - \big(y(z)/e'(z)\big) > \alpha\big(e'(w)/e'(w)\big) - \big(y(z)/e'(z)\big) \\ &= \alpha - \big(y(z)/e'(z)\big) \geqslant \alpha - \alpha\big(e'(z)/e'(z)\big) = 0. \end{aligned}$$

We now show that our conclusion above holds for an arbitrary positive element of E'. Let $y \in B(e')$, $y \geqslant 0$, $y \notin I(e')$, and suppose that y is not a constant. We know that $y = \sup_n\{\wedge ne'\}$. Suppose, for every $u \in E$, $u \neq 0$, with $e'(u) = 0$, $y(u) = 0$. Then consequently $y \wedge ne' = 0$ for every $n \in \mathbb{N}$, and so all $y \wedge ne'$ are constants, since otherwise being positive elements of $I(e')$, by the previous conclusion, there must exist a $u \in E$, with $e'(u) = 0$ for which $(y \wedge ne')(u) > 0$ for every $n \in \mathbb{N}$ and that contradicts our assumption about y. So all $y \wedge ne'$ must be constants. To complete this proof, it remains to show that, if all $y \wedge ne'$ are constants, so is y. Set $y \wedge e' = \alpha e'$ with $\alpha_1 \in \mathbb{R}^+$,

$$\begin{aligned} &y \wedge 2e' = \alpha_2 e' \quad \text{with } \alpha_2 \in \mathbb{R}^+ \text{ and } \alpha_1 > \alpha_2, \\ &\quad\vdots \\ &y \wedge ne' = \alpha_n e' \quad \text{with } \alpha_n \in \mathbb{R}^+ \text{ and } \alpha_n > \cdots > \alpha_2 > \alpha_1, \\ &\quad\vdots \end{aligned}$$

Then $(\alpha_n e')_{n\in\mathbb{N}}$ is an increasing sequence. From $\sup_n\{y \wedge ne'\} = y$, it follows that $\sup_n\{\alpha_n e'\} = y$. It is then clear that $\sup_n\{\alpha_n\|e'\|\} \leqslant \|y\|$. So $\sup_n\{\alpha_n\} \leqslant (\|y\|/\|e'\|)$ and the sequence $(\alpha_n)_{n\in\mathbb{N}}$ converges. Hence $y = \lim_{n\to\infty} \alpha_n e' = e' \lim_{n\to\infty} \alpha_n = \alpha_\infty e'$ with $\alpha_\infty \in \mathbb{R}^+$. Thus y is a constant. □

The next theorem discusses a generalization of Theorem 2.1 in [2] to the case when T is a power-bounded positive operator in a *KB*-space.

Theorem 1.4. *Let E be a KB-space with a weak-order unit e and let E' be the dual of E with a weak order unit e'. Let $T : E \to E$ with $Te \leqslant e$ be a power-bounded positive operator and let $T' : E' \to E'$ with $T'e' = e'$ be its adjoint. Then the following conditions are equivalent*:

(a) *T' is mixing.*
(b) *All weak*-limit points in E of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in I(e')$.*
(c) *For every $u \in (e)$ with $e'(u) = 0$, $\lim_{n\to\infty} x(T^n u) = 0$ for every $x \in I(e')$.*
(d) *For every $v \in I(e)$, $\lim_{N\to\infty} \|N^{-1}\sum_{i=1}^N T^{n_i} v - (e'(v)/e'(e))e\|_E = 0$ for every increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$.*

Proof. By Theorem 1.2, (d) $\Rightarrow$ (a). A careful inspection of the proof of Theorem 1.1 shows that (a) implies (b) and (b) implies (c) as well in this setting.

(c) $\Rightarrow$ (b) Here we adapt the ideas in the proof of [2, Theorem 2.1, p. 232]. Let y be a weak*-limit point in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ for $x \in E'$, $x \geqslant 0$. If $y \geqslant 0$ and y is not a constant then, by Proposition 1.2, there must exist an element u of E with $e'(u) = 0$ and $y(u) > 0$. Since y is a weak*-limit point of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ there exists an increasing subsequence $(n_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty}\langle u, T'^{n_i} x\rangle = \lim_{i\to\infty}\langle T^{n_i} u, x\rangle = \langle u, y\rangle > 0$. But that contradicts (c). Hence y has to be a constant. Let x be an arbitrary element of E'. Then from the decomposition $x = x^+ - x^-$ it follows that the latter conclusion remains true for every $x \in E'$.

(a) $\Rightarrow$ (d) Assume (a). Then (c) follows at once. We claim that condition (c) can be extended to the entire E'. Let $x \in E'$, $u \in I(e)$, $e'(u) = 0$, and $\varepsilon > 0$. E has an order-continuous norm; therefore there exits $y \in I(e')$ such that $\|x - y\|'_E < \varepsilon/2K$, where $K > 0$, $K \geqslant \sup_n\{\|T^n u\|_E\}$. Obviously, we have

$$\left|x(T^n u)\right| \leqslant \left|(x-y)(T^n u)\right| + \left|y(T^n u)\right| \leqslant \|x-y\|_{E'}\|T^n u\|_E + \left|y(T^n u)\right|.$$

It is clear that $\|x - y\|_{E'}\|T^n u\|_E \leqslant \varepsilon/2$. By condition (c), n can be chosen large enough to get $|y(T^n u)| < \varepsilon/2$. Hence, for $u \in E$, $e'(u) = 0$, $(T^n u)_{n\in\mathbb{N}\cup\{0\}}$ converges to 0 weakly. From the latter extension of (c), one obtains that all weak*-limit points in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in E'$, by repeating the proof of the implication (c) $\Rightarrow$ (b). The equivalence of (b) with (c) allows us to get $Te = e$, by proceeding as in Step 1 of the proof of Lemma 1.1, and so (d) follows by Theorem 1.2.

To complete this proof we have to establish that (b) $\Rightarrow$ (a). To this end, assume (b). By $Te \leqslant e$ and $T'e' = e'$, as well as the equivalence of (b) with (c), we obtain $Te = e$ as in the proof of the preceding implication. Then (b) implies (a) by Theorem 1.1. □

Definition 2.1 (see [1, p. 284]). Let $T : X \to Y$ be an operator between two Banach spaces. Then T is said to be weakly compact whenever T carries the closed unit ball of X onto a relatively weakly compact subset of Y.

This important result is due to Eberlin and Šmulian.

Theorem 2.1 (see [1, p. 156]). *A subset of a normed space X is relatively weakly compact if and only if every sequence of A has a subsequence that converges weakly to some element of X.*

For weakly compact positive operators, Theorem 1.4 can be extended to Banach lattices more general than *KB*-spaces. To achieve this goal we need the following results.

Lemma 2.1. *Let E be a Banach lattice with a weak-order unit e and an order-continuous norm. Let E' be the dual of E with a weak-order unit e'. Let $T : E \to E$ with $Te \leqslant e$ be a power-bounded operator and let $T' : E' \to E'$ with $T'e' = e'$ be its adjoint. Suppose T is weakly compact and, for every $u \in E$ with $e'(u) = 0$, $\lim_{n\to\infty} x(T^n u) = 0$ for every $x \in E'$. Then there exist an increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$, and a constant z in E such that* $\lim_{i\to\infty}\langle |T^{n_i} u| - z, x\rangle = 0$, $\forall x \in E'$.

Proof. By Theorem 2.1, T is weakly compact if and only if for every norm-bounded sequence $(x_n)_{n\in\mathbb{N}}$ of E the sequence $(Tx_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in E. Therefore the subset $\{T|T^n u|: n = 0, 1, \ldots\}$ is relatively weakly compact and so one can prove the lemma by carefully repeating the proof of Lemma 1.1. □

Theorem 2.2. *Let E, E', and T' be as in Lemma* 2.1. *Let $T : E \to E$ be a power-bounded positive operator which is also weakly compact. If all weak*-limit points in E' of $(T'^n x)_{n\in\mathbb{N}\cup\{0\}}$ are constants for every $x \in E'$ and T has a non-zero* (*positive*) *fixed point then, for every $v \in I(e)$,*

$$\lim_{N\to\infty}\left\| N^{-1}\sum_{i=1}^{N} T^{n_i} v - \bigl(e'(v)/e'(e)\bigr)e \right\|_E = 0$$

for every increasing subsequence of natural numbers $(n_i)_{i\in\mathbb{N}}$.

Proof. Combine Theorem 1.3 and Lemma 2.1. □

Theorem 2.3. *Let E be a Banach lattice with a weak-order unit e and an order-continuous norm and let E' be its dual with a weak-order unit e'. Let $T : E \to E$ with $Te \leqslant e$ be a weakly compact power-bounded positive operator and let $T' : E' \to E'$ with $T'e' = e'$ be the adjoint of T. Then, as in Theorem* 1.4, *conditions* (a), (b), (c), *and* (d) *are equivalent.*

Proof. By Lemma 2.1 and Theorem 2.2, the proof of this theorem can be read in between the lines of the proof of Theorem 1.4. □

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